APPENDIX

Raudenbush and Willms (1995) wrote down a fairly general linear model for value-added, then used a very simple model for illustration. The illustration is informative for this brief. We have an outcome Y_{ij}, a linear function of a covariate X_{ij} defined for each student *i* within classroom *j* $(i = 1,\ldots,n_j; j = 1,\ldots J)$. We can orthogonally decompose the regression within and between classrooms as

$$Y_{ij} = \beta_0 + \beta_w(X_{ij} - \bar{X}_{\cdot j}) + \beta_b(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot}) + U_j + e_{ij}$$

(A1)

where $\bar{X}_{\cdot j}$ is the classroom mean of the covariate, $\bar{X}_{\cdot\cdot}$ is the overall ("grand") mean; β_w characterizes the association between X and Y within classrooms; and β_b characterizes the association between and the classroom mean of the covariate and the classroom mean outcome. By re-centering the covariate, we obtain a re-parameterization of (A1) known in the sociology literature as the "contextual effects model."

$$Y_{ij} = \beta_0 + \beta_w(X_{ij} - \bar{X}_{\cdot\cdot}) + \beta_c(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot}) + U_j + e_{ij}$$

(A2)

Here $\beta_c = \beta_b - \beta_w$ is the "contextual effect" (the association between $\bar{X}_{\cdot j}$ and the mean outcome controlling for person-level X_{ij}); U_j and e_{ij} are random effects at the classroom and student levels respectively; and we assume

Ignorable student-level error $E(e_{ij} \mid X,) = E(e) = 0$ Assumption (i)

We allow for confounding between, but do not, however, assume $E(U_j \mid \bar{X}) = E(U_j) = 0$. Let us now average the outcome within classrooms to obtain

$$\bar{Y}_{\cdot j} = \beta_0 + \beta_w(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot}) + \beta_c(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot}) + U_j + \bar{e}_{\cdot j}$$

(A3)

Type A and Type B Effects

Now we can define two effects. The first is the Type A effect, of interest to parents choosing classrooms, that is

$$\beta_c(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot}) + U_j$$

This is composed of two components. The first is the contextual component $\beta_c(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot})$, attributable to the composition of the classroom with respect to the covariate $\bar{X}_{\cdot j}$. This might represent the impact of peer composition, the quality of the resources available to the teacher, or the supportiveness of the parents. The second is the random effect U_j, attributable to the effectiveness of teacher *j*'s practice.

The Type B effect, of interest to teacher evaluators, is thus the random effect U_j itself. That is,

$$B_j \equiv U_j$$

We shall assume that this key estimand – the true value added for teacher j, has zero mean and variance

$$Var(U_j) = \tau^2$$

(A4)

Identification

Under Assumption (i) in connection with A2, we can identify β_w, and hence the Type A effect by subtraction

$$A_j \approx \bar{Y}_{\cdot j} - \beta_0 - \beta_w(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot}) = \beta_c(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot}) + U_j + \bar{e}_{\cdot j}$$
$$= A_j + \bar{e}_{\cdot j}$$

(A5)

Equation (A5) is what economists would call the estimated "fixed effect."

Under an additional strong identification assumption $Cov(U_j, \bar{X}_{\cdot j}) = 0$, we can identify the Type B effect by subtraction as

$$B_j \approx \bar{Y}_{\cdot j} - \beta_0 - \beta_w(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot}) - \beta_c(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot})$$
$$= \bar{Y}_{\cdot j} - \beta_0 - \beta_b(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot}) = U_j + \bar{e}_{\cdot j}$$

(A6)

However, if $Cov(U_j, \bar{X}._j) \neq 0$, we will estimate β_b with bias, that is, the least squares estimator $\hat{\beta}_b$ will have expectation

$$E(\hat{\beta}_b) = \beta_b + \beta_{u\bar{x}}$$
(A7)

where $\beta_{u\bar{x}} = Cov(U_j, \bar{X}._j)/\sigma^2_{\bar{x}}$ and $\sigma^2_{\bar{x}}$ is the variance of the sample mean of the covariate $\bar{X}._j$ (Note this variance will be non-zero even if n_j is infinite, assuming classrooms vary in composition). As a result, if $Cov(U_j, \bar{X}._j) \neq 0$, (6) does not approximate the true type B effect but instead estimates a quantity we might called the "prima facie" effect, that is

$$B_{PFj} = U_j - \beta_{u\bar{x}}(\bar{X}._j - \bar{X}..)$$
(A8)

A Way Forward

We can put a *bound* on how biased our analysis will be using the following result Raudenbush and Willms (1995):

$$Var(B_{PFj}) \leq Var(B_j) \leq Var(A_j)$$
$$or$$
$$(1-\rho^2_{u\bar{x}})\tau^2 \leq \tau^2 \leq (\beta_c + \beta_{u\bar{x}})^2\sigma^2_{\bar{x}} + (1-\rho^2_{u\bar{x}})\tau^2$$
(A9)

We can consistently estimate $Var(B_{PFj})$ and, if Assumption (i) is plausible, we can consistently estimate $Var(A_j)$. We can therefore put bounds around the variance of the true value-added effects. Moreover, it follows from (A9) that the variance of the bias is

$$0 \leq \rho^2_{u\bar{x}}\tau^2 \leq (\beta_c + \beta_{u\bar{x}})^2\sigma^2_{\bar{x}}$$
(A10)

If (A5) is reasonable, we can estimate the upper bound as

$$(\beta_c + \beta_{u\bar{x}})^2\sigma^2_{\bar{x}} \approx \hat{\beta}^2_c\sigma^2_{\bar{x}}$$
(A11)

If the estimated contextual effect is null or if the classroom mean $\overline{X}_{\cdot j}$ varies little, we need not worry. If their product is large, our biases are large in magnitude. We might then back away from the universal "Type B" effect and instead block on classrooms with similar $\overline{X}_{\cdot j}$. This would narrow the scope of the reference population for teacher *j* to include other teachers whose classrooms have similar composition.

Note schools are not represented in our model. Hence the Type B effect is in fact the sum of the teacher value-added plus the effectiveness of the school.